

Fischer–Muszély functional equation almost everywhere

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Dedicated to Professor János Aczél on his 90th birthday

Abstract. We show that, under suitable assumptions, a function f from a group $(G, +)$ into a real or complex inner product space $(H, (\cdot|\cdot))$, satisfying the Fischer–Muszély functional equation

$$\|f(x+y)\| = \|f(x) + f(y)\|$$

for all pairs (x, y) off a sufficiently small (negligible) subset of G^2 has to be almost everywhere equal to an additive map from G into H , i.e. the set $\{x \in G : f(x) \neq a(x)\}$ is small (negligible) in G . Small sets in G and G^2 are defined in an axiomatic way. Several corollaries illustrating some consequences of this result are presented.

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In the last half-century the functional equation

$$(FM) \quad \|f(x+y)\| = \|f(x) + f(y)\|$$

was attracting much attention from numerous mathematicians. It has extensively been studied, among others, by Fischer and Muszély [6] (see also an earlier version [5] in Hungarian, and [4]), Dhombres [3], Aczél and Dhombres [1], Berruti and Skof [2], Skof [15], Ger [9], [10], Ger and Kocłęga [12] and Schöpf [14], who presented various partial results for mappings with values in some special Banach spaces like inner product spaces, strictly convex spaces or in general normed linear spaces but with scalar domains. In [11] the general solution of the equation mentioned was given. J. Tabor Jr. has shown in [16] that the Fischer–Muszély equation is Hyers–Ulam stable in the class of surjective mappings.

The main goal of the present paper is to exhibit another stability property: we shall show that under suitable assumptions a function satisfying the

Fischer–Muszély functional equation postulated almost everywhere has to coincide with an additive map almost everywhere.

In what follows the symbol $(G, +)$ will stand for an additively written group although commutativity is not assumed. Recall that a nonempty family $\mathcal{J} \subset 2^G \setminus \{G\}$ is called a *proper linearly invariant ideal* (briefly: p.l.i. ideal) in G provided that it satisfies the following conditions:

- (i) if $A, B \in \mathcal{J}$, then $A \cup B \in \mathcal{J}$;
- (ii) if $A \in \mathcal{J}$ and $B \subset A$, then $B \in \mathcal{J}$;
- (iii) if $A \in \mathcal{J}$ and $x \in G$, then $x - G \in \mathcal{J}$.

We say that a property $\mathcal{P}(x)$ holds \mathcal{J} -almost everywhere in G whenever $\mathcal{P}(x)$ is valid for all $x \in G \setminus U$ for some set $U \in \mathcal{J}$.

For a subset $M \subset G^2$ and $x \in G$ we define a *section*

$$M[x] := \{y \in G : (x, y) \in M\}.$$

An ideal $\widehat{\mathcal{J}}$ in G^2 is said to be *conjugate* with an ideal \mathcal{J} in G if and only if for every set $M \in \widehat{\mathcal{J}}$ the appurtenance $M[x] \in \mathcal{J}$ takes place \mathcal{J} -almost everywhere in G .

The family

$$\Omega(\mathcal{J}) := \{M \subset G^2 : M[x] \in \mathcal{J} \text{ for } \mathcal{J}\text{-almost all } x \in G\}$$

yields the largest (in the sense of set inclusion) p.l.i. ideal in G^2 being conjugate to \mathcal{J} (see Ger [7] or Kuczma [13, Ch. XVII, §5]).

Our main result reads as follows.

Theorem. *Given a p.l.i. ideal \mathcal{J} in a group $(G, +)$ and a real or complex inner product space $(H, (\cdot|\cdot))$, assume that a map $f : G \rightarrow H$ satisfies the equation*

$$(FM) \quad \|f(x+y)\| = \|f(x) + f(y)\|$$

for all pairs $(x, y) \in G^2$ off a set $M \in \Omega(\mathcal{J})$ such that $T_1(M)$ and $T_2(M)$ stay in $\Omega(\mathcal{J})$ for $T_1(x, y) := (y, x)$ and $T_2(x, y) := (y, x - y)$, $(x, y) \in G^2$.

If, moreover, for any set U from \mathcal{J} the set $\frac{1}{2}U := \{x \in G : 2x \in U\}$ belongs to \mathcal{J} and there exists a member E of \mathcal{J} such that

$$M \cap \bigcup_{k=1}^3 \{(x, kx) \in G^2 : x \notin E\} = \emptyset,$$

then there exists a unique additive map $a : G \rightarrow H$ such that

$$\{x \in G : f(x) \neq a(x)\} \in \mathcal{J}.$$

Proof. To apply the technique used by Fischer and Muszély in [6] (see also Aczél and Dhombres [1, p. 139]), fix arbitrarily an $x \in G \setminus (E \cup \frac{1}{2}E)$; then all the pairs (x, x) , $(x, 2x)$ and $(x, 3x)$ as well as $(2x, 2x)$ are off M and we have

$$\begin{aligned} \|f(2x)\| &= 2\|f(x)\|, \quad \|f(3x)\| = \|f(x) + f(2x)\|, \\ 4\|f(x)\| &= \|f(4x)\| = \|f(x) + f(3x)\|, \end{aligned}$$

which like in [7], forces the equality

$$f(2x) = 2f(x) \quad \text{to be valid for all } x \in G \setminus (E \cup \tfrac{1}{2}E). \quad (1)$$

Since M is supposed to be a member of $\Omega(\mathcal{J})$, there exists a set $U \in \mathcal{J}$ such that for every $x \in G \setminus U$ the section $M[x]$ falls into \mathcal{J} .

Let N stand for the set-theoretical union of the following seven sets: M , $(E \cup (\frac{1}{2}E)) \times G$, $G \times (E \cup (\frac{1}{2}E))$ and

$$M_1 := \{(x, y) \in G^2 : x \in \tfrac{1}{2}U \text{ or } y \in M[2x]\},$$

$$M_2 := \{(x, y) \in G^2 : x \in U \text{ or } y \in \tfrac{1}{2}M[x]\},$$

$$M_3 := \{(x, y) \in G^2 : x \in U \text{ or } y \in -x + M[x]\}, \quad M_4 := (T_1 \circ T_2)(M).$$

Each one of these seven sets yields a member of the ideal $\Omega(\mathcal{J})$. Indeed, this is obvious for the first three sets as well as, by the invariance assumptions, for the set M_4 . To check that $M_1 \in \Omega(\mathcal{J})$ note that for every $x \notin \frac{1}{2}U \in \mathcal{J}$ the section

$$\begin{aligned} M_1[x] &= \{y \in G : (x, y) \in M_1\} = \{y \in G : y \in M[2x]\} \\ &= M[2x] \quad \text{belongs to } \mathcal{J}. \end{aligned}$$

Similarly, since for every $x \notin U \in \mathcal{J}$ the section

$$\begin{aligned} M_2[x] &= \{y \in G : (x, y) \in M_2\} = \{y \in G : y \in \tfrac{1}{2}M[2x]\} \\ &= \tfrac{1}{2}M[2x] \quad \text{belongs to } \mathcal{J}, \end{aligned}$$

we infer that $M_2 \in \Omega(\mathcal{J})$. Finally, for every $x \notin U \in \mathcal{J}$ the section

$$\begin{aligned} M_3[x] &= \{y \in G : (x, y) \in M_3\} = \{y \in G : y \in -x + M[x]\} \\ &= -x + M[x] \quad \text{belongs to } \mathcal{J}, \end{aligned}$$

which shows that $M_3 \in \Omega(\mathcal{J})$.

Consequently, the union N of all the sets spoken of yields a member of the ideal $\Omega(\mathcal{J})$ as well. Now, fix arbitrarily a pair $(x, y) \in G^2 \setminus N$. Then:

1. $\|f(x+y)\| = \|f(x) + f(y)\|$ because $(x, y) \notin M$;
2. $f(2x) = 2f(x)$ and $f(2y) = 2f(y)$ because of (1) and the fact that $x, y \notin E \cup \frac{1}{2}E$;
3. $\|f(2x+y)\| = \|f(2x) + f(y)\|$ because $(x, y) \notin M_1$ which forces the pair $(2x, y)$ to stay off the set M ;
4. $\|f(2x+y)\| = \|f(x) + f(x+y)\|$ because $(x, y) \notin M_3$ which forces the pair $(x, x+y)$ to stay off the set M ;
5. $\|f(x+2y)\| = \|f(x) + f(2y)\|$ because $(x, y) \notin M_2$ which forces the pair $(x, 2y)$ to stay off the set M ;

6. $\|f(x + 2y)\| = \|f(x + y) + f(y)\|$ because $(x, y) \notin M_4$ which forces the pair $(x + y, y)$ to stay off the set M ;

Relations 3. and 4. jointly with 2. imply that

$$\|f(x) + (f(x) + f(y))\| = \|f(x) + f(x + y)\|, \quad (2)$$

whereas a similar conclusion

$$\|(f(x) + f(y)) + f(y)\| = \|f(x + y) + f(y)\|, \quad (3)$$

can be drawn from relations 5. and 6. jointly with 2. By means of 1., after squaring both sides of (2) and (3), by a simple calculation, we derive the equalities

$$\Re((f(x)|f(x + y) - f(x) - f(y))) = 0 = \Re((f(y)|f(x + y) - f(x) - f(y))),$$

respectively, which immediately imply that

$$\Re((f(x) + f(y)|f(x + y) - f(x) - f(y))) = 0. \quad (4)$$

Along the same lines as in the paper [6] of P. Fischer and Gy. Muszély, from the trivial equality

$$\|f(x + y)\|^2 = \|(f(x) + f(y)) + (f(x + y) - f(x) - f(y))\|^2$$

with the aid of 1. and (4) we derive the relationship

$$\|f(x + y) - f(x) - f(y)\|^2 = 0.$$

This clearly forces the additivity relation

$$f(x + y) = f(x) + f(y)$$

that remains valid for all pairs $(x, y) \in G^2 \setminus N$, i.e. $\Omega(\mathcal{J})$ -almost everywhere in G^2 . Now, it remains to apply a de Bruijn's type result from [8]: there exists a unique additive function $a : G \rightarrow H$ such that the equality $f(x) = a(x)$ holds for \mathcal{J} -almost all $x \in G$, i.e.

$$\{x \in G : f(x) \neq a(x)\} \in \mathcal{J}.$$

Thus the proof has been completed. \square

Remark 1. The leading idea of the proof above was to run along the lines of the proof presented in [6] treating it as the obstacle race. However, the set of obstacles, although basically caused by the fact that the validity of the (FM) equation is postulated merely almost everywhere, was enlarged by another one; namely, close to the bottom of page 199 in [6] the authors write:

If we interchange the variables x and y in the equation (16) we get

$$(17) \quad \operatorname{Re}[f(y), f(x + y) - (f(x) + f(y))] = 0,$$

which is wrong; actually, we get then

$$\operatorname{Re}[f(y), f(y + x) - (f(x) + f(y))] = 0,$$

and not (17) because of the lack of the commutativity of the domain semigroup.

In what follows we shall present a few corollaries illustrating some consequences of the theorem just proved.

Corollary 1. *Let $(X, \|\cdot\|)$ stand for a normed linear space and let $(H, (\cdot|\cdot))$ be an inner product space. If a map $f : X \rightarrow H$ satisfies the Fischer–Muszély functional equation (FM) in a vicinity of infinity (outside an arbitrarily given ball centered at the origin), then there exists a unique additive map $a : X \rightarrow H$ and a bounded set $B \subset X$ such that $f(x) = a(x)$ for all $x \in X \setminus B$.*

Proof. Let \mathcal{J} stand for the p.l.i. ideal of all bounded subsets of the space X . Clearly, any bounded set and, in particular, any ball $M := B((0, 0), r)$ in the product space X^2 yields a member of $\Omega(\mathcal{J})$. Assume that

$$\|f(x+y)\| = \|f(x) + f(y)\| \quad \text{for all pairs } (x, y) \in X^2 \setminus M.$$

Put $T_1(x, y) := (y, x)$ and $T_2(x, y) := (y, x - y)$, $(x, y) \in X^2$. The images $T_1(M)$ and $T_2(M)$ are contained in M and $\sqrt{5}M$, respectively, so that they stay in $\Omega(\mathcal{J})$. Moreover, $\frac{1}{2}U$ is bounded for any bounded set U . Finally, since the set $E := \{x \in X : \|x\| \leq r\}$ belongs to \mathcal{J} and for every $x \in X \setminus E$ one has $\|(x, kx)\| = \sqrt{1+k^2}\|x\| \geq \sqrt{2}r > r$ for $k \in \{1, 2, 3\}$, the condition

$$M \cap \bigcup_{k=1}^3 \{(x, kx) \in G^2 : x \notin E\} = \emptyset$$

is satisfied. Thus all the assumptions of the Theorem are fulfilled which ends the proof. \square

Corollary 2. *Let $(G, +)$ stand for a uniquely 2-divisible locally compact group and let $(H, (\cdot|\cdot))$ be an inner product space. Denote by h_1 and h_2 the left Haar measures in G and G^2 , respectively, with $h_1(G) = \infty$; moreover, let h_1^* be the outer Haar measure associated with h_1 . Assume that for every set $U \subset G$ one has $h^*(\{x \in G : 2x \in U\}) < \infty$ provided that $h^*(U) < \infty$. If a map $f : G \rightarrow H$ satisfies the Fischer–Muszély functional equation (FM) for all $(x, y) \in G^2 \setminus M$ where $M \subset G^2$ is a set of finite measure h_2 and such that*

$$M \cap \bigcup_{k=1}^3 \{(x, kx) \in G^2 : x \in G\} = \emptyset,$$

then there exists a unique additive map $a : G \rightarrow H$ and a set $B \subset G$ such that $h_1^(B) < \infty$ and $f(x) = a(x)$ for all $x \in G \setminus B$.*

Proof. Let \mathcal{J} stand for the p.l.i. ideal of all subsets of G having finite outer measure h_1^* . Since, by Fubini's theorem, one has

$$\infty > h_2(M) = \int_G h_1(M[x]) dh_1(x),$$

we infer that h_1 -almost all sections $M[x]$ are of finite h_1 measure. This proves that M falls into the ideal $\in \Omega(\mathcal{J})$. Let T_1 and T_2 be defined as in the statement

of the Theorem. Directly from the definition of the product measure it follows that $h_2(T_1(M)) = h_2(M) < \infty$ and

$$\begin{aligned} h_2(T_2(M)) &= \int_G h_1(T_2(M)[x]) dh_1(x) = \int_G h_1(-x + T_1(M)[x]) dh_1(x) \\ &= \int_G h_1(T_1(M)[x]) dh_1(x) = h_2(T_1(M)) = h_2(M) < \infty. \end{aligned}$$

Therefore, h_1 -almost all sections $T_2(M)[x]$ are of finite h_1 measure which forces the image $T_2(M)$ to fall into the ideal $\Omega(\mathcal{J})$. To finish the proof it suffices to apply the Theorem. \square

Corollary 3. *Let $(G, +)$ stand for a uniquely 2-divisible Polish topological group and let $(H, (\cdot|\cdot))$ be an inner product space. Assume that the map $G \ni x \mapsto \frac{1}{2}x \in G$ is a homeomorphism of G onto itself. If a map $f : G \rightarrow H$ satisfies the Fischer–Muszély functional equation (FM) for all $(x, y) \in G^2 \setminus M$ where $M \subset G^2$ is a first category (in the sense of Baire) subset of the group G^2 and such that*

$$M \cap \bigcup_{k=1}^3 \{(x, kx) \in G^2 : x \in G\} = \emptyset,$$

then there exists a unique additive map $a : G \rightarrow H$ and a first category set $B \subset G$ such and $f(x) = a(x)$ for all $x \in G \setminus B$.

Proof. Let \mathcal{J} stand for the p.l.i. ideal of all first category sets in G . Then with the aid of the celebrated Kuratowski–Ulam theorem we establish the fact that M belongs to the ideal $\Omega(\mathcal{J})$. Since the maps $T_1(x, y) := (y, x)$ and $T_2(x, y) := (y, x - y)$, $(x, y) \in G^2$ yield homeomorphic selfmappings of G^2 we infer that both the images $T_1(M)$ and $T_2(M)$ stay in $\Omega(\mathcal{J})$. Moreover since, by assumption, the map $G \ni x \mapsto \frac{1}{2}x \in G$ is a homeomorphism of G onto itself, the set $\frac{1}{2}U$ is of the first Baire category provided that so is U . To finish the proof it remains to apply the Theorem. \square

Corollary 4. *Let $(\mathbb{Z}, +)$ be the additive group of all integers and let $(H, (\cdot|\cdot))$ be an inner product space. If a sequence $(a_n)_{n \in \mathbb{Z}}$ of elements of the space H satisfies the Fischer–Muszély equation*

$$\|a_{n+m}\| = \|a_n + a_m\| \tag{5}$$

for all but finite set of pairs $(n, m) \in \mathbb{Z}^2$, then there exists a unique vector $c \in H$ such that $a_n = nc$ for all but finite number of integers n .

Proof. Let \mathcal{J} stand for the p.l.i. ideal of all finite subsets of \mathbb{Z} . Assuming that relation (5) holds for all $n, m \in \mathbb{Z}$ off a set $M := \{(n, m) \in \mathbb{Z}^2 : |n|, |m| \leq n_0\}$ where n_0 is a positive integer, we see that M belongs to the ideal $\Omega(\mathcal{J})$. Plainly the maps $T_1(n, m) := (m, n)$ and $T_2(n, m) := (m, n - m)$, $(n, m) \in \mathbb{Z}^2$ transform finite sets into finite sets, which forces the images $T_1(M)$ and $T_2(M)$

to stay in $\Omega(\mathcal{J})$. Moreover, for every finite set $U \subset \mathbb{Z}$ the set $\{n \in \mathbb{Z} : 2n \in U\}$ is finite as well. Finally, on setting $E := \{-n_0, \dots, -1, 0, 1, \dots, n_0\}$ we have $E \in \mathcal{J}$ and M is disjoint with the union

$$\bigcup_{k=1}^3 \{(n, kn) \in \mathbb{Z}^2 : n \notin E\}$$

that is contained in $\mathbb{Z}^2 \setminus M$. Thus all the assumptions of the Theorem are fulfilled which implies the existence of a unique additive map $a : \mathbb{Z} \rightarrow H$ such that the set $\{n \in \mathbb{Z} : a(n) \neq a_n\}$ is finite. Since, obviously, $a(n) = na(1)$, $n \in \mathbb{Z}$, we get the equality $a_n = nc$ for all but a finite number of integers n , with a unique $c := a(1) \in H$, as claimed. \square

Remark 2. As it is, the formulation of the Theorem leaves room for improvements. For instance, it would be desirable to have:

- the group considered replaced by a semigroup;
- the inner product space replaced by a strictly convex one;
- the assumption

$$M \cap \bigcup_{k=1}^3 \{(x, kx) \in G^2 : x \notin E\} = \emptyset,$$

removed.

Unfortunately, at present none of these three wishes can be accomplished because of the proof technique applied.

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References

- [1] Aczél, J., Dhombres, J.: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
- [2] Berruti, G., Skof, F.: Risultati di equivalenza per un'equazione di Cauchy alternativa negli spazi normati. Atti Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **125**(5-6), 154–167 (1991)
- [3] Dhombres, J.: Some Aspects of Functional Equations. Chulalongkorn Univ, Bangkok (1979)
- [4] Fischer, P.: Remarque 5–Probleme 23. Aequationes Math. **1**, 300 (1968)
- [5] Fischer, P., Muszély, G.: A Cauchy-féle függvényegyenletek bizonyos típusú általánosításai. Mat. Lapok **16**, 67–75 (1965)
- [6] Fischer, P., Muszély, G.: On some new generalizations of the functional equation of Cauchy. Can. Math. Bull. **10**, 197–205 (1967)
- [7] Ger, R.: On some functional equations with a restricted domain, I., II. Fund. Math. **89**, 131–149 (1975); **98**, 249–272 (1978)
- [8] Ger, R.: Note on almost additive functions. Aequationes Math. **17**, 73–76 (1978)

- [9] Ger, R.: On a characterization of strictly convex spaces. *Atti Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **127**, 131–138 (1993)
- [10] Ger, R.: Fischer–Muszély additivity of mappings between normed spaces, The First Katowice-Debrecen Winter Seminar on Functional Equations, Report of Meeting. *Ann. Math. Sil.* **15**, 89–90 (2001)
- [11] Ger, R.: Fischer–Muszély additivity on Abelian groups, *Comment. Math., Tomus Specialis in Honorem Juliani Musielak*, 83–96 (2004)
- [12] Ger, R., Kocłęga, B.: Isometries and a generalized Cauchy equation. *Aequationes Math.* **60**, 72–79 (2000)
- [13] Kuczma, M.: *An Introduction to the Theory of Functional Equations and Inequalities*. Polish Scientific Publishers & Silesian University, Warszawa (1985)
- [14] Schöpf, P.: Solutions of $\|f(\xi + \eta)\| = \|f(\xi) + f(\eta)\|$. *Math. Pannon.* **8/1**, 117–127 (1997)
- [15] Skof, F.: On the functional equation $\|f(x + y) - f(x)\| = \|f(y)\|$. *Atti Acad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **127**, 229–237 (1993)
- [16] Tabor, J. Jr.: Stability of the Fischer–Muszély functional equation. *Publ. Math. Debrecen* **62**, 205–211 (2003)

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